THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 3

§20) 2) b) For $f(z) = (2z^2+i)^5$, by Chain rule,

$$f'(z) = \frac{d(2z^2+i)^5}{d(2z^2+i)}\frac{d(2z^2+i)}{dz} = 5(2z^2+i)^4(4z) = 20z(z^2+i)^4.$$

§20) 3) a) Since for all $n \in \mathbb{N}$, $a_n z^n$ is differentiable with $\frac{d}{dz}a_n z^n = na_n z^{n-1}$, we have

$$\begin{aligned} P'(z) &= \frac{d}{dz}(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n) \\ &= \frac{d}{dz}(a_0) + \frac{d}{dz}(a_1 z) + \frac{d}{dz}(a_2 z^2) + \cdots + \frac{d}{dz}(a_n z^n) \\ &= a_1 + 2a_2 z + \cdots + na_n z^{n-1}. \end{aligned}$$

b) For any $k = 0, 1, 2, \ldots, n$, note that

$$P^{(k)}(z) = k(k-1)\ldots(1)a_k + (k+1)(k)\ldots(2)a_{k+1}z + \cdots + (n)(n-1)\ldots(n-k+1)a_n z^{n-k}.$$

Hence, $P^{(k)}(0) = k(k-1)\ldots(1)a_k$ and $a_k = \frac{P^{(k)}(0)}{k!}$.

§24) 1) b) Note that $f(z) = z - \bar{z} = 2yi$. Hence we have $u(x,y) = 0$ and $v(x,y) = 2y$.

Since $u_x = 0 \neq 2 = v_y$ for any $z \in \mathbb{C}$, $f(z)$ does note satisfy the Cauchy-Riemann equations and thus is not differentiable everywhere.

c) Note that $f(z) = 2x + ixy^2$. Hence we have $u(x,y) = 2x$ and $v(x,y) = xy^2$.

Note that

$$\begin{aligned} u_x = v_y &\implies 2 = 2xy \implies xy = 1 \\ u_y = -v_x &\implies 0 = y^2 \implies y = 0 \end{aligned}$$

When $y = 0$, $xy = 0 \neq 1$. Therefore, $f(z)$ does note satisfy the Cauchy-Riemann equations for any $z \in \mathbb{C}$ and thus is not differentiable everywhere.

§24) 2) a) For $f(z) = iz + 2 = (2-y) + ix$, we have $u(x,y) = 2-y$ and $v(x,y) = x$. Note that u and v are differentiable for any $z \in \mathbb{C}$. Since $u_x = 0 = v_y$ and $u_y = -1 = -v_x$, $f(z)$ is differentaible everywhere with

$$f'(z) = u_x + iv_x = i.$$

Similarly, for $f'(z) = i = a(x,y) + b(x,y)$, we have $a(x,y) = 0$ and $b(x,y) = 1$. Note that a and b are differentaible for any $z \in \mathbb{C}$. Since $a_x = 0 = b_y$ and $a_y = 0 = -b_x$, $f'(z)$ is differentaible everywhere with

$$f''(z) = 0.$$

d) For $f(z) = \cos x \cosh y - i \sin x \sinh y$, we have $u(x,y) = \cos x \cosh y$ and $v(x,y) = -\sin x \sinh y$. Note that u and v are differentiable for any $z \in \mathbb{C}$. Since $u_x = -\sin x \cosh y = v_y$ and $u_y = \cos x \sinh y = -v_x$, $f(z)$ is differentaible everywhere with

$$f'(z) = u_x + iv_x = -\sin x \cosh y - i \cos x \sinh y.$$

Similarly, for $f'(z) = -\sin x \cosh y - i \cos x \sinh y = a(x,y) + b(x,y)$, we have $a(x,y) = -\sin x \cosh y$ and $b(x,y) = -\cos x \sinh y$. Note that a and b are differentaible for any $z \in \mathbb{C}$. Since $a_x = -\cos x \cosh y = b_y$ and $a_y = -\sin x \sinh y = -b_x$, $f'(z)$ is differentaible everywhere with

$$f''(z) = a_x + ib_x = -\cos x \cosh y + i \sin x \sinh y.$$

§24) 4) b) For $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$, we have $u(r,\theta) = e^{-\theta}\cos(\ln r)$ and $v(r,\theta) = e^{-\theta}\sin(\ln r)$. Note that u and v are differentiable for any $r > 0, \theta \in (0, 2\pi)$. Since $u_r = -\dfrac{e^{-\theta}\sin(\ln r)}{r} = \dfrac{1}{r}v_\theta$ and $\dfrac{1}{r}u_\theta = -\dfrac{e^{-\theta}\cos(\ln r)}{r} = -v_r$, $f(z)$ is differentaible everywhere with

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) \\ &= e^{-i\theta}\frac{-e^{-\theta}\sin(\ln r) + ie^{-\theta}\cos(\ln r)}{r} \\ &= i\frac{e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)}{re^{i\theta}} \\ &= i\frac{f(z)}{z}. \end{aligned}$$

§24) 5) Note that $u_r = u_x\cos\theta + u_y\sin\theta$ and $u_\theta = -u_x r\sin\theta + u_y r\cos\theta$. Note that

$$\begin{aligned} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} \\ \implies \begin{pmatrix} u_x \\ u_y \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}^{-1} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} \\ &= \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} \\ &= \begin{pmatrix} u_r\cos\theta - u_\theta\dfrac{\sin\theta}{r} \\ u_r\sin\theta + u_\theta\dfrac{\cos\theta}{r} \end{pmatrix}. \end{aligned}$$

Similarly, we have

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_r\cos\theta - v_\theta\dfrac{\sin\theta}{r} \\ v_r\sin\theta + v_\theta\dfrac{\cos\theta}{r} \end{pmatrix}.$$

Note that if $ru_r = v_\theta$ and $u_\theta = -rv_r$, then

$$\begin{aligned} u_x = u_r\cos\theta - u_\theta\frac{\sin\theta}{r} &= v_\theta\frac{\cos\theta}{r} + v_r\sin\theta = v_y, \\ u_y = u_r\sin\theta + u_\theta\frac{\cos\theta}{r} &= v_\theta\frac{\sin\theta}{r} - v_r\cos\theta = -v_x. \end{aligned}$$

Thus, $ru_r = v_\theta$ and $u_\theta = -rv_r$ is the CR equations in polar form.

§24) 6) From § 24) 5), we have

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r},$$
$$v_x = v_r \cos\theta - v_\theta \frac{\sin\theta}{r}.$$

If $f(z)$ is differentiable at a nonzero point $z_0 = r_0 \exp(i\theta_0)$, we have $ru_r = v_\theta$ and $u_\theta = -rv_r$.

Furthermore,

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= u_r \cos\theta - u_\theta \frac{\sin\theta}{r} + i(v_r \cos\theta - v_\theta \frac{\sin\theta}{r}) \\ &= u_r \cos\theta + v_r \sin\theta + i(v_r \cos\theta - u_r \sin\theta) \\ &= (\cos\theta - i \sin\theta)(u_r + iv_r) \\ &= e^{-i\theta}(u_r + iv_r), \end{aligned}$$

where u_r and v_r are evaluated at (r_0, θ_0).

§26) 4) c) For $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$, note that

$$(z+2)(z^2+2z+2) = 0$$
$$\implies z = -2 \text{ or } z = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{2}i}{2} = -1 \pm i.$$

As a result, the singular points of $f(z)$ are given by $-2, -1+i$ and $-1-i$.

Since outside the singular points, $p(z) = z^2 + 1$ and $q(z) = (z+2)(z^2+2z+2)$ are analyic with $q(z) \neq 0$, the function $f(z) = \dfrac{p(z)}{q(z)}$ is analyic outside the singular points.